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# AN ILLUMINATION PROBLEM FOR ZONOIDS

BY

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ABSTRACT

Let $\mathbf{Z} \subset \mathbf{E}^d$ be a $d$-dimensional zonoid, where $d \geq 3$. Boltjanskii and Soltan recently proved that if $\mathbf{Z}$ is not a parallelotope, then $\mathbf{Z}$ can be illuminated by $3 \cdot 2^{d-2}$ points disjoint from $\mathbf{Z}$. In the present paper we prove a related result. Namely, we show that if $d+1=2^p$, then $\mathbf{Z}$ can be illuminated by $\frac{2^d}{d+1}$ lines lying outside $\mathbf{Z}$.

## 1. Introduction

Let $\mathbf{E}^d$ denote a $d$-dimensional Euclidean space, where $d \geq 1$. A convex body in $\mathbf{E}^d$ is a compact convex set with non-empty interior. Let $\mathbf{K} \subset \mathbf{E}^d$ be a convex body. We say that a point $L \in \mathbf{E}^d \setminus \mathbf{K}$ illuminates the boundary point $P$ of $\mathbf{K}$ if the open ray emanating from $P$ having direction vector $\overrightarrow{LP}$ has a non-empty intersection with the interior of $\mathbf{K}$. Furthermore, we say that the points $\{L_1, L_2, \ldots, L_n\} \subset \mathbf{E}^d \setminus \mathbf{K}$ illuminate $\mathbf{K}$ if every boundary point of $\mathbf{K}$ is illuminated by at least one of the points $L_1, L_2, \ldots, L_n$. Finally, let $I_0(\mathbf{K})$ be the smallest number of points lying outside $\mathbf{K}$ which illuminate $\mathbf{K}$. A zonotope of

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$\mathbf{E}^d$ is a convex polytope that represents the vector sum of finitely many closed segments in $\mathbf{E}^d$. A compact convex set in $\mathbf{E}^d$ is called a zonoid if it is the limit (in the sense of the Hausdorff metric) of some convergent sequence of zonotopes of $\mathbf{E}^d$ [6]. Boltjanskii and Soltan [5] recently proved the following nice theorem on zonoids. If the convex body $\mathbf{Z} \subset \mathbf{E}^d$ is a zonoid other than a parallelotope, then $I_0(\mathbf{Z}) \leq 3 \cdot 2^{d-2}$, where $d \geq 3$. In the present paper we prove a related result on zonoids.

The following general illumination problem was raised and partially discussed in [1], [2], [3] and [4]. Let $\mathbf{K} \subsetneq \mathbf{E}^d$ be a $d$-dimensional closed convex set, where $d \geq 1$. We say that the affine subspace $L \subset \mathbf{E}^d \setminus \mathbf{K}$ of dimension $0 \leq \dim L \leq d-1$ illuminates the boundary point $P$ of $\mathbf{K}$ if there exists a point of $L$ which illuminates $P$. Furthermore, we say that the affine subspaces $L_1, L_2, \ldots, L_n \subset \mathbf{E}^d \setminus \mathbf{K}$ illuminate $\mathbf{K}$ if every boundary point of $\mathbf{K}$ is illuminated by at least one of the affine subspaces $L_1, L_2, \ldots, L_n$. Finally, let $I_l(\mathbf{K})$ be the smallest cardinality of affine subspaces of dimension $l$ lying in $\mathbf{E}^d \setminus \mathbf{K}$ which illuminate $\mathbf{K}$, where $0 \leq l \leq d-1$. Obviously,

$$1 \leq I_{d-1}(\mathbf{K}) \leq I_{d-2}(\mathbf{K}) \leq \cdots \leq I_1(\mathbf{K}) \leq I_0(\mathbf{K}).$$

We prove the following

THEOREM: *Let $\mathbf{Z} \subset \mathbf{E}^d$ be a $d$-dimensional zonoid and $\mathbf{C} \subset \mathbf{E}^d$ be a $d$-dimensional parallelotope. If $d+1 = 2^p$, then*

$$I_1(\mathbf{Z}) \leq I_1(\mathbf{C}) = 2^{2^p-p-1} = \frac{2^d}{d+1}, \quad \text{where } p \geq 2.$$

*Remark:* The proof of Lemma 1 below shows that the pure combinatorial equivalent of $I_1(\mathbf{C}) = 2^d/d+1$ is the existence of a perfect covering of the vertices of the edge graph of the $d$–dimensional parallelotope $\mathbf{C} \subset \mathbf{E}^d$ by $2^d/d+1$ shortest paths of length $d$, where $d+1 = 2^p$ and $p \geq 2$.

Generalizing the question of the Theorem we pose the following

CONJECTURE: *Let $\mathbf{K} \subset \mathbf{E}^d$ be a convex body, where $d \geq 2$. Then*

$$I_1(\mathbf{K}) \leq \left\lceil \frac{2^d}{d+1} \right\rceil.$$

A $d$-dimensional closed convex set $\mathbf{K} \subsetneq \mathbf{E}^d$ is called almost bounded if there exists a $d$-dimensional ball of $\mathbf{E}^d$ which intersects every supporting hyperplane of $\mathbf{K}$, where $d \geq 1$. If $\mathbf{K}$ is almost bounded, then let $L$ denote the closed convex cone which is the union of closed half-lines emanating from an interior point, say, $O$ of $\mathbf{K}$ and lying in $\mathbf{K}$. Moreover, let $Pr_\perp : \mathbf{E}^d \to L^\perp$ denote the orthogonal projection of $\mathbf{E}^d$ onto the affine subspace $O \in L^\perp$ which is the orthogonal complement of the affine hull of $L$ in $\mathbf{E}^d$ and let $I_l[\mathrm{cl}(Pr_\perp(\mathbf{K}))]$ denote the corresponding illumination number of the closure $\mathrm{cl}(Pr_\perp(\mathbf{K}))$ of $Pr_\perp(\mathbf{K})$ in $L^\perp$, where $0 \leq l \leq d-1$. If $\dim L^\perp \leq l$, then we take $I_l[\mathrm{cl}(Pr_\perp(\mathbf{K}))] = 1$. The following theorem was proved in [4]. Let $\mathbf{K} \subsetneq \mathbf{E}^d$ be a $d$-dimensional almost bounded closed convex set and let $0 \leq l \leq d-1$. Then $\mathrm{cl}(Pr_\perp(\mathbf{K}))$ is a compact convex set of dimension $\dim L^\perp$ and $I_l(\mathbf{K}) \leq I_l[\mathrm{cl}(Pr_\perp(\mathbf{K}))] < +\infty$. Combining this result with our Theorem one can get the following

COROLLARY: *Let $\mathbf{K} \subsetneq \mathbf{E}^d$ be a d-dimensional almost bounded closed convex set and assume that $cl(Pr_\perp(\mathbf{K}))$ is a zonoid with $\dim\,[cl(Pr_\perp(\mathbf{K}))] + 1 = 2^p$. Then $I_1(\mathbf{K}) \leq 2^{2^p-p-1}$.*

## 2. Proof of Theorem

First we verify the following special case.

LEMMA 1: *Let $\mathbf{C}_d \subset \mathbf{E}^d$ be a d-dimensional cube and assume that $d+1 = 2^p$, where $p \geq 2$. Then*

$$I_1(\mathbf{C}_d) = 2^{2^p-p-1} = \frac{2^d}{d+1}.$$

*Proof:* Without loss of generality we may assume that the vertex set of $\mathbf{C}_d$ is the vector space $[GF(2)]^d$. In the edge graph of $\mathbf{C}_d$ two vertices $x$ and $y$ of $\mathbf{C}_d$ are connected by an edge if and only if $x+y$ has exactly one non-zero component. Let

$$e_i \stackrel{\text{def}}{=} (0, \ldots, 0, \overset{i}{1}, 0, \ldots, 0) \in [GF(2)]^d,$$

where $1 \leq i \leq d$. It is easy to see that if $v$ is a vertex of $\mathbf{C}_d$, then the vertices $v, v+e_1, v+e_1+e_2, \ldots, v+e_1+e_2+\cdots+e_d$ can be illuminated by a line of $\mathbf{E}^d \setminus \mathbf{C}_d$. Thus, in order to show that

$$I_1(\mathbf{C}_d) \leq \frac{2^d}{d+1}, \qquad \text{where } d+1 = 2^p \text{ and } p \geq 2$$

it is sufficient to prove that there are vertices $v_1, v_2, \ldots, v_n$ of $\mathbf{C}_d$ with $n = 2^{2^p-p-1}$ such that the paths

$$\{v_i, v_i+e_1, v_i+e_1+e_2, \ldots, v_i+e_1+e_2+\cdots+e_d\}$$

cover all vertices of $\mathbf{C}_d$, where $1 \le i \le n$. The proof is by induction on $p$. As the claim is obviously true for $p = 2$ assume that for $d' = 2^{p-1} - 1$ and $p \ge 3$ there exists a set $\{v'_1, v'_2, \dots, v'_{n'}\}$ of $n' = 2^{2^{p-1}-p}$ vertices of $\mathbf{C}_{d'}$ such that the paths

$$\{v'_i, v'_i + e'_1, v'_i + e'_1 + e'_2, \dots, v'_i + e'_1 + e'_2 + \dots + e'_{d'}\}$$

cover all vertices of $\mathbf{C}_{d'}$, where

$$e'_j \stackrel{\text{def}}{=} (0, \dots, 0, \overset{j}{1}, 0, \dots, 0) \in [GF(2)]^{d'} \quad \text{and} \quad 1 \le j \le d'.$$

Then take the set

$$\{(x, 0, x + v'_i) \in [GF(2)]^d | x \in [GF(2)]^{d'} \text{ and } 1 \le i \le n'\}$$

of $2^{d'} \cdot n' = 2^{2^{p-1}-1} \cdot 2^{2^{p-1}-p} = 2^{2^p-p-1} = n$ vertices of $\mathbf{C}_d$. Now we have to show that the above $n$ paths of the edge graph of $\mathbf{C}_d$ starting from the vertices $(x, 0, x + v'_i)$ are pairwise disjoint. Thus, assume that

$$(1) \qquad (x, 0, x + v'_i) + \sum_{k=1}^{l} e_k = (y, 0, y + v'_j) + \sum_{k=1}^{m} e_k$$

for some $x, y \in [GF(2)]^{d'}$, $1 \le i, j \le n'$, $1 \le l, m \le d$. Without loss of generality we may assume that $l \le m$. From (1) we get

$$(2) \quad (x+y, \overset{d'+1}{0}, x+y+v'_i+v'_j) = \sum_{k=1}^{l} e_k + \sum_{k=1}^{m} e_k = (0, \dots, 0, \overset{l+1}{1}, \dots, \overset{m}{1}, 0, \dots, 0).$$

Thus, either $d' + 1 \le l \le m$ or $l \le m \le d'$. In the first case we easily get that $x = y$ and

$$(0, \dots, \overset{d'+1}{0}, v'_i + v'_j) = \sum_{k=1}^{l} e_k + \sum_{k=1}^{m} e_k$$

i.e.

$$(0, \dots, \overset{d'+1}{0}, v'_i) + \sum_{k=1}^{l} e_k = (0, \dots, \overset{d'+1}{0}, v'_j) + \sum_{k=1}^{m} e_k$$

which then by induction implies that $i = j$ and $l = m$. In the second case we get that $x + y = \sum_{k=1}^{l} e'_k + \sum_{k=1}^{m} e'_k$ and $x + y = v'_i + v'_j$. Hence, $v'_i + v'_j =$

$\sum_{k=1}^{l} e'_k + \sum_{k=1}^{m} e'_k$, that is $v'_i + \sum_{k=1}^{l} e'_k = v'_j + \sum_{k=1}^{m} e'_k$ which then again by induction implies that $i = j$, $l = m$ and so $x = y$. This completes the proof of

$$I_1(\mathbf{C}_d) \leq \frac{2^d}{d+1} \text{ for } d+1 = 2^p \text{ and } p \geq 2.$$

Finally, it is easy to see that a line of $\mathbf{E}^d \setminus \mathbf{C}_d$ can illuminate at most $d+1$ vertices of $\mathbf{C}^d$. Thus,

$$I_1(\mathbf{C}^d) = \frac{2^d}{d+1},$$

indeed. ∎

As the illumination numbers are affine invariants Lemma 1 extends to $d$-dimensional parallelotopes as well. Now we are in a position to prove the claim of the Theorem for zonotopes.

Lemma 2: *Let $\mathbf{P} \subset \mathbf{E}^d$ be a $d$-dimensional zonotope and assume that $d+1 = 2^p$, where $p \geq 2$. Then*

$$I_1(\mathbf{P}) \leq 2^{2^p-p-1} = \frac{2^d}{d+1}.$$

*Proof:* Recall the following separation lemma of [1] and [2]. Let $L \subset \mathbf{E}^d \setminus \{O\}$ be an affine subspace of dimension $0 \leq l \leq d-1$, where $O$ denotes the origin of $\mathbf{E}^d$. Then let

$$\hat{L} = \cap\{H_Q | H_Q = \{X \in \mathbf{E}^d | \langle \overrightarrow{OX}, \overrightarrow{OQ} \rangle = 1\} \text{ and } Q \in L\},$$

where $\langle\ ,\ \rangle$ denotes the usual inner product of $\mathbf{E}^d$. It is easy to see that $\dim \hat{L} = d - l - 1$. Then the separation lemma can be formulated as follows.

Proposition: *Let $\mathbf{K}$ be a convex body of $\mathbf{E}^d$ that contains the origin $O$ as an interior point and let $F_m$ be the smallest dimensional face of $\mathbf{K}$ which contains the boundary point $P$ of $\mathbf{K}$, where $d \geq 1$. Then the affine subspace $L \subset \mathbf{E}^d \setminus \mathbf{K}$ of dimension $0 \leq \dim L \leq d-1$ illuminates $P$ if and only if there exists $Q \in L$ such that the hyperplane*

$$H_Q = \{X \in \mathbf{E}^d | \langle \overrightarrow{OX}, \overrightarrow{OQ} \rangle = 1\} \supset \hat{L}$$

*strictly separates $O$ from the face*

$$F_m^* = \{X \in \mathbf{K}^* | \langle \overrightarrow{OX}, \overrightarrow{OY} \rangle = 1 \text{ for all } Y \in F_m\}$$

*of the polar convex body*

$$\mathbf{K}^* = \{X \in \mathbf{E}^d | \langle \overrightarrow{OX}, \overrightarrow{OY} \rangle \leq 1 \text{ for all } Y \in \mathbf{K}\}.$$

*Furthermore, $I_l(\mathbf{K}) = n$ if and only if $n$ is the smallest integer such that there exist affine subspaces $\hat{L}_1, \hat{L}_2, \dots, \hat{L}_n$ of $\mathbf{E}^d$ of dimension $d-l-1$ with the property that every face of the polar convex body $\mathbf{K}^*$ can be strictly separated from $O$ by a hyperplane of $\mathbf{E}^d$ which contains at least one of the affine subspaces $\hat{L}_1, \hat{L}_2, \dots, \hat{L}_n$.*

Without loss of generality we may assume that the origin $O$ of $\mathbf{E}^d$ is the center point of the $d-$dimensional zonotope $\mathbf{P} \subset \mathbf{E}^d$. Let us consider the polar convex polytope

$$\mathbf{P}^* = \{X \in \mathbf{E}^d | \langle \overrightarrow{OX}, \overrightarrow{OY} \rangle \leq 1 \text{ for all } Y \in \mathbf{P}\}$$

of $\mathbf{P}$. $O$ is the center point of $\mathbf{P}^*$ as well. Let $\mathbf{S}^{d-1}$ be a $(d-1)$-dimensional sphere centered at $O$ which lies in the interior of $\mathbf{P}^*$. Then it is easy to prove that the central projection of the faces of $\mathbf{P}^*$ from $O$ onto $\mathbf{S}^{d-1}$ is a tiling $\mathcal{T}$ of $\mathbf{S}^{d-1}$ which can be obtained as a dissection of $\mathbf{S}^{d-1}$ by finitely many, say, $n$ $(d-2)$-dimensional great spheres. The Proposition implies that it is sufficient to prove that there are $2^d/d+1$ $(d-2)$-dimensional affine subspaces of $\mathbf{E}^d$ with the property that every face of $\mathcal{T}$ can be strictly separated from $O$ by a hyperplane of $\mathbf{E}^d$ which contains at least one of the $2^d/d+1$ $(d-2)$-dimensional affine subspaces. It is clear that $n \geq d$ and there are $d$ affinely independent $(d-2)$-dimensional great spheres among the $n$ ones such that the dissection $\mathcal{T}^1$ of $\mathbf{S}^{d-1}$ generated by them is the central projection of the faces of a $d$-dimensional affine crosspolytope $\mathbf{C}^*$ of $\mathbf{E}^d$ from the center point $O$ onto $\mathbf{S}^{d-1}$. As the polar convex body $\mathbf{C}$ of $\mathbf{C}^*$ is a $d$-dimensional parallelotope i.e. an affine image of a $d$-dimensional cube and as the affinity does not change the illumination number $I_1(\mathbf{K})$ of any convex body $\mathbf{K}$, Lemma 1 and the Proposition imply that there exist $\frac{2^d}{d+1}$ $(d-2)$-dimensional affine subspaces of $\mathbf{E}^d$ with the property that every face of the tiling $\mathcal{T}^1$ can be strictly separated from $O$ by a hyperplane of $\mathbf{E}^d$ which contains at least one of the $2^d/d+1$ $(d-2)$-dimensional affine subspaces. Then it remains to observe the rather trivial fact that the same $2^d/d+1$ $(d-2)$-dimensional affine subspaces of $\mathbf{E}^d$ posses the property that every face of the tiling $\mathcal{T}$ can be strictly separated from $O$ by a hyperplane of $\mathbf{E}^d$ which contains at least one of the $2^d/d+1$ $(d-2)$-dimensional affine subspaces. This completes the proof of Lemma 2. ∎

The following two lemmas are due to Boltjanskii and Soltan [5] in case $l = 0$. As the proofs of the following slightly more general lemmas can be obtained as a rather trivial extensions of Boltjanskii's and Soltan's methods we omit the details here.

LEMMA 3: *Let $0 \leq l \leq d-1$ be integers. Then on the class of all convex bodies in $\mathbf{E}^d$ the function $I_l(\mathbf{K})$ is upper semi continuous i.e. if the sequence of convex bodies $\mathbf{K}_1, \mathbf{K}_2, \ldots, \mathbf{K}_m, \ldots$ converges to $\mathbf{K}$, and $\mathbf{K}$ is a convex body, then*

$$I_l(\mathbf{K}) \geq \overline{\lim_{m\to+\infty}} I_l(\mathbf{K}_m).$$

It is known that a compact convex set $\mathbf{Z} \subset \mathbf{E}^d$ is a zonoid if and only if (up to a parallel translation) it represents the set of all points $x(g) = \int_0^t g(s)\varphi'(s)ds$, where $x = \varphi(s)$, $0 \leq s \leq t$, is the vector equation of some rectifiable curve in $\mathbf{E}^d$ on which the parameter $s$ is the length, and $g$ runs through the set of measurable functions satisfying the condition $|g(s)| \leq \frac{1}{2}$ with $0 \leq s \leq t$. Moreover, let $0 \leq s_1 < s_2 < \cdots < s_k \leq t$ be points at which the derivative $\varphi'(s)$ exists and is approximately continuous. Then the zonotope i.e. the vector sum of the $k$ closed intervals respectively parallel to the vectors $\varphi'(s_1), \varphi'(s_2), \ldots, \varphi'(s_k)$ is called a tangential zonotope of the zonoid $\mathbf{Z}$ (see [5]).

LEMMA 4: *Let $\mathbf{Z} \subset \mathbf{E}^d$ be a $d$-dimensional zonoid and $\mathbf{P}$ be a $d$-dimensional tangential zonotope of it moreover, let $0 \leq l \leq d-1$. Then $I_l(\mathbf{Z}) \leq I_l(\mathbf{P})$.*

According to a result of Baladze (see [5]) every $d$-dimensional zonoid $\mathbf{Z} \subset \mathbf{E}^d$ can be represented as the limit of some sequence of its $d$-dimensional tangential zonotopes. This and Lemma 3 and 4 then imply that if $\mathbf{Z} \subset \mathbf{E}^d$ is a $d$-dimensional zonoid and $0 \leq l \leq d-1$, then there exists a sequence $\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_m, \ldots$ of its $d$-dimensional tangential zonotopes such that $I_l(\mathbf{Z}) = \lim_{m\to+\infty} I_l(\mathbf{P}_m)$. Thus, Lemma 2 immediately yields that if $d+1 = 2^p$, where $p \geq 2$, then

$$I_1(\mathbf{Z}) = \lim_{m\to+\infty} I_1(\mathbf{P}_m) \leq 2^{2^p-p-1} = \frac{2^d}{d+1}.$$

This completes the proof of the Theorem. ∎

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